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Differential quadrature application in post-buckling analysis of a hinged-fixed elastica under terminal forces and self-weight †

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Abstract

Based on geometrically non-linear theory for extensible elastic rods, governing equations of statically post-buckling of a beam with one end hinged and the other fixed, and subjected to a terminal force and a self-weight, are established. The formulation is derived from geometrical compatibility, equilibrium of forces and moments, and constitutive relations, which characterize a complex two-point boundary value problem. By using differential quadrature method (DQM), the non-linear governing equations are solved numerically and the post-buckled configurations of the deformed column are presented. Results are plotted in non-dimensional graphs for a range of density and terminal force, and are in good agreement with available references.

Keywords: Differential quadrature (DQ); Non-linear geometry; Post-buckling; Heavy column; Large deformation

1. Introduction

In conventional column buckling problems, the self-weight is often neglected since it is assumed to be small when compared to the applied axial loads. When the column self-weight is relatively significant, it has to be taken into consideration in the buckling analysis and such column is generally referred to as a "heavy" column [1]. Grischcoff [2] obtained the buckling load for the combined effect of both the self-weight and an axial load for a cantilevered column. This line of investigation was later extended by Wang and Drachman [3,4] to include the case of a finite column hanging from its foundation. In addition, Wang and Ang [5] studied the stability of heavy column from the energy approach. In their study, the simple formulas to various column end conditions for buckling capacity are obtained. Vaz et al. [6, 7] performed the post-buckling analysis of slender elastic rods subjected to terminal forces and self-weight using the shooting method. Li and Zhou [8] presented the post-buckling of a hinged-fixed beam under a pair of following forces via the shooting method.

Differential quadrature method, as a powerful and efficient numerical method, is used in the analysis of large deformation problems in the works of Malekzadeh et al. [9-11]. In addition, Karami and Malekzadeh [12] constructed a new differential quadrature methodology for beam analysis as well as through the associated differential quadrature element method. Many illustrative numerical examples are shown in the works of Malekzadeh et al. on the efficiency, accuracy, and CPU time requirement of DQM in solid mechanics. Malekzadeh [13] presented the comparison of convergence and accuracy of the first four frequencies between the DQM and both orthogonal polynomial-Ritz method and the Chebyshev-Ritz method. Malekzadeh and Vosoughi [9] showed the comparison of CPU time requirement for the evaluation of the first three nonlinear frequencies between the DQM and Finite Element Method (FEM). Results show that CPU time requirement of the DQM may decrease to hundredth (%1) in comparison to other methods such as FEM and Finite Difference Method (FDM), with sound agreement in accuracy and convergence.

This paper deals with the post-buckling analysis of a hinged-fixed elastica under terminal forces and self-weight via differential quadrature method. Post-buckling configurations and some characteristic curves are plotted in good agreement with available references.

2. Mathematical formulation

Fig. 1 shows a heavy column of length L, Young's modulus E, second moment of area I, and weight per unit length q, which is subjected to an axial load P. A reference Cartesian coordinate with x−axis coinciding with the neutral axis of the un-deformed column, and y−axis aligned in direction of the

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off-axis deformation was used. Let $C\ (x,0)$ with $x \in [0\ ,\ L]$ denote a point of the pre-deformed axial line of the column, where x and y form a Cartesian coordinate system. When the rod is in a buckled state, the material point C moves to point $C'\ (x+u,\ w)$, in which u and w are the displacements of point C in x and y directions, respectively. Here, it is presumed that the deformed central axis is still in the Oxy plane.

Based on the geometrically nonlinear theory of axially extensible elastic rods, the basic equations can be expressed as

Geometric equations:

$$\frac{ds}{dx}=R\ ,\ \frac{du}{dx}=R\cos\theta-1\ ,\ \frac{dw}{dx}=R\sin\theta \tag{1}$$

$$\varepsilon=\frac{ds-dx}{dx}=R-1\ ,\ \kappa=\frac{1}{R}\frac{d\theta}{dx} \tag{2}$$

Equilibrium equations:

$$\begin{aligned}&N(x)=-P\cos\theta+H\sin\theta-\int_0^x q(\eta)d\eta\cos\theta\\&M(x)=-Pw(x)-H(x+u-u_0)+\\&\int_0^x q(\eta)d\eta.(w(x)-w(\eta))\end{aligned} \tag{3}$$

Constitutive equations:

$$N=AE(R-1),\ M=EI\frac{d\theta}{dx} \tag{4}$$

where $s\ (x)$, $R\ (x)$, $\theta\ (x)$, $\varepsilon\ (x)$, and $\kappa\ (x)$ are the arc length of the deflection curve, stretch ratio of the axial line, rotation angle of the cross-section, strain of the axial line, and curvature of the deflection curve, respectively. In addition, $N\ (x)$, $M\ (x)$, P and H are the axial internal force, internal bending moment, terminal force at the hinged end along the x direction and horizontal constraint force, respectively. The equality $\int_0^{s(x)} q'(\xi)d\xi=\int_0^x q(\eta)d\eta$ has been employed in the above derivation, where $q'(\xi)$ is the distributed vertical load contributed by the weight on the deformed column and $q(\eta)$ is the distributed load on Cartesian coordinate system with $0\le\xi\le s$ and $0\le\eta\le x$. Bending moment and stretch ratio expressions are obtained using Eqs. (3) and (4) as

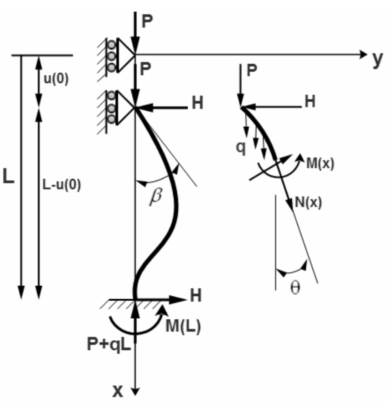


Fig. 1. Hinged-fixed heavy column under axial forces P and self-weight.

$$\begin{aligned}&EI\frac{d\theta}{dx}=-Pw(x)-H(x+u-u_0)+\\&\int_0^x q(\eta)d\eta.(w(x)-w(\eta))\\&R=1-\frac{P}{AE}\cos\theta+\frac{H}{AE}\sin\theta-(\int_0^x q(\eta)d\eta)\frac{\cos\theta}{AE}\end{aligned} \tag{5}$$

Differentiating Eq. (5) with respect to x yields:

$$\begin{aligned}&EI\frac{d^2\theta}{dx^2}+PR\sin\theta+HR\cos\theta+\\&(\int_0^x q(\eta)d\eta)R\sin\theta=0\end{aligned} \tag{6}$$

with the boundary conditions of a hinged-fixed beam

$$\begin{aligned}&w=0\ ,\ \frac{d\theta}{dx}=0 \qquad at\ x=0\\&w=0\ ,\ \theta=0\ ,u=0 \quad at\ x=L\end{aligned} \tag{7}$$

by introducing the following dimensionless quantities

$$(\bar{x},\bar{s},\bar{u},\bar{w})=\frac{(x,s,u,w)}{L},\ (\bar{P},\bar{H},\bar{q})=\frac{(P,H,qL)L^2}{EI} \tag{8}$$

Consequently, the governing equations in dimensionless form can be written as

$$\begin{aligned}&\frac{d^2\theta}{d\bar{x}^2}+\bar{P}R\sin\theta+\bar{H}R\cos\theta+\int_0^{\bar{x}}\bar{q}(\eta)d\eta)R\sin\theta=0\\&R=1-\frac{1}{\lambda^2}[\bar{P}\cos\theta-\bar{H}\sin\theta+(\int_0^{\bar{x}}\bar{q}(\eta)d\eta)\cos\theta]\\&\bar{w}\Big|^{\bar{x}=0,1}=0,\ \frac{d\theta}{d\bar{x}}\Big|^{\bar{x}=0}=0,\ \bar{u}\Big|^{\bar{x}=1}=0,\ \theta\Big|^{\bar{x}=1}=0\end{aligned} \tag{9}$$

where $\lambda^2=AL^2/I$ and $\lambda=120$ is used for comparison and verification with other reference results.

3. The initial buckling solution

Due to the strong nonlinearity of the governing equations, it is difficult to find any analytical solution. Thus, the differential quadrature method is used for solving the nonlinear equations. In the first step, the initial buckling solution is constructed via DQM. In the case of combined loads (terminal force P and column weight q), each load has a contribution in buckling occurrence, which will be investigated in this section.

Now, by applying DQM, Eq. (9) is discretized at n-4 inner points, which produces n-4 equations for buckling analysis. Henceforth, the buckling loads will depend on the applied boundary conditions. In ordinary polynomial-based differential quadrature, the kth-order derivative of the solution function w at grid point i in one dimension can be written as [14]:

$$w_i^{(k)}=\sum_{j=1}^{n}c_{ij}^{(k)}w_j \quad (i=1,2,...,n)$$
$$c_{ij}^{(1)}=\frac{M_n(x_i)}{(x_j-x_i)M_n(x_j)} \quad (i\neq j)$$
$$c_{ii}^{(1)}=\sum_{j=1,i\neq j}^{n}\frac{1}{(x_i-x_j)},M_n(x_i)=\prod_{j=1,j\neq i}^{n}(x_i-x_j) \tag{10}$$
$$c_{ij}^{(m)}=m\left(c^{(1)}{}_{ij}c_{ii}^{(m-1)}-\frac{c_{ij}^{(m-1)}}{(x_i-x_j)}\right), m=2,3,...,n-1$$
$$c_{ii}^{(m)}=-\sum_{j=1,j\neq i}^{n}c_{ij}^{(m)}$$

where $c_{ij}^{(k)}$ is the weighting coefficients of kth-order derivative, and n and w_j are the total number of grid points including the two boundary points, and the solution values at grid point j, respectively. Also, the following non-uniform grid spacing is used for fast convergence, namely [14]

$$x(i)=\left(\frac{1}{2}\right)\left[1-\cos\pi\frac{(i-1)}{(n-1)}\right] \quad (i=1,2,...,n) \tag{11}$$

The following assumption is used to calculate the initial buckling load

$$R\cos\theta\sim 1\ ,\ R\sin\theta\sim\theta\ ,\ \theta=\frac{d\bar{w}}{d\bar{x}} \tag{12}$$

Finally, dimensionless forms of the governing equations for buckling analysis are written as

$$\frac{d^4\bar{w}}{d\bar{x}^4}+\bar{P}\frac{d^2\bar{w}}{d\bar{x}^2}+\bar{q}\,\bar{x}\frac{d^2\bar{w}}{d\bar{x}^2}+\bar{q}\frac{d\bar{w}}{d\bar{x}}=0$$
$$\bar{w}\Big|^{\bar{x}=0,1}=0,\ \frac{d^2\bar{w}}{d\bar{x}^2}\Big|^{\bar{x}=0}=0,\ \frac{d\bar{w}}{d\bar{x}}\Big|^{\bar{x}=1}=0 \tag{13}$$

The DQ discretized form of the resulting equations can be expressed as

$$\sum_{j=1}^{n}c_{ij}^{(4)}\bar{w}_j+\bar{P}\sum_{j=1}^{n}c_{ij}^{(2)}\bar{w}_j+\bar{q}\,(\sum_{j=1}^{n}c_{ij}^{(2)}\bar{w}_j\bar{x}_i$$
$$+\sum_{j=1}^{n}c_{ij}^{(1)}\bar{w}_j)=0\ ,\quad 3\leq i\leq n-2$$
$$\bar{w}(1)=\bar{w}(n)=0 \tag{14}$$
$$\sum_{j=1}^{n}c_{ij}^{(2)}\bar{w}_j\Big|^{i=1}=o,\ \sum_{j=1}^{n}c_{ij}^{(1)}\bar{w}_j\Big|^{i=n}=0$$

The above equations consist a set of linear equations used for calculation of initial buckling loads, denoted by $\bar{P}_{cr}^{*}$ ($\bar{P}\neq 0,\bar{q}=0$), $\bar{q}_{cr}^{*}$ ($\bar{q}\neq 0,\bar{P}=0$) and $\bar{P}_{cr}\ \&\ \bar{q}_{cr}$ ($\bar{P}\neq 0$, $\bar{q}\neq 0$). The above set of linear equations can be expressed in compact form as

Table 1. The values of critical weight $\bar{q}_{cr}^{*}$.

Method	Present	[5]	[8]	[1]
$\bar{q}_{cr}^{*}$	52.6799	53.91	52.666	52.5007
Error (%)	-	2.33	0.02	0.34

Table 2. The values of critical terminal load $\bar{P}_{cr}$, for some prescribed values of $\bar{q}$.

Method	Buckling load	Difference (%)	Buckling load
Present [5]	$\bar{P}_{cr}$ = 16.600 $\bar{P}_{cr}$ = 16.662	- 0.37	$\bar{q}_{cr}$ = 10 $\bar{q}_{cr}$ = 10
Present [5]	$\bar{P}_{cr}$ = 14.841 $\bar{P}_{cr}$ = 14.849	- 0.05	$\bar{q}_{cr}$ = 15 $\bar{q}_{cr}$ = 15
Present [5]	$\bar{P}_{cr}$ = 12.899 $\bar{P}_{cr}$ = 12.997	- 0.78	$\bar{q}_{cr}$ = 20 $\bar{q}_{cr}$ = 20
Present [5]	$\bar{P}_{cr}$ = 11.166 $\bar{P}_{cr}$ = 11.108	- 0.52	$\bar{q}_{cr}$ = 25 $\bar{q}_{cr}$ = 25
Present [5]	$\bar{P}_{cr}$ = 9.187 $\bar{P}_{cr}$ = 9.180	- 0.07	$\bar{q}_{cr}$ = 30 $\bar{q}_{cr}$ = 30
Present [5]	$\bar{P}_{cr}$ = 7.228 $\bar{P}_{cr}$ = 7.212	- 0.22	$\bar{q}_{cr}$ = 35 $\bar{q}_{cr}$ = 35

$$[A]_{(n\times n)}\{\theta\}_{(n)}=\lambda\,[B]_{(n\times n)}\{\theta\}_{(n)} \tag{15}$$

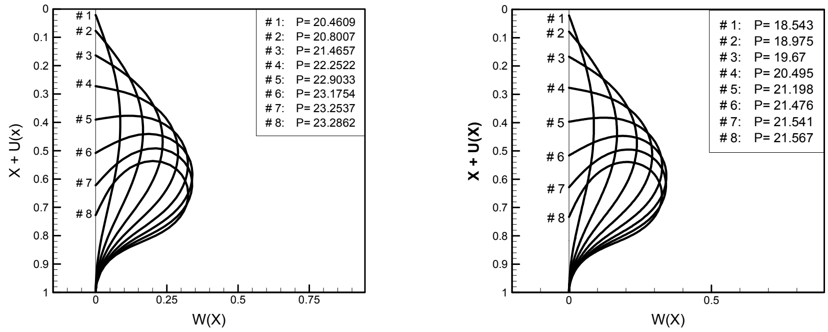
3.1 Numerical results

A FORTRAN code is developed based on the differential quadrature method in order to solve the eigen value problem. The results are compared with available references with good agreement (Tables 1, 2), though the DQM has simpler assumptions and less computation burden. The column stability behavior and its post-buckling configuration are influenced by the value of column weight (Table 2). In the absence of terminal force, the column remains stable up to a critical value of density ($\bar{q}\leq 52.6799$) and unstable beyond that.

4. Post-buckling numerical results

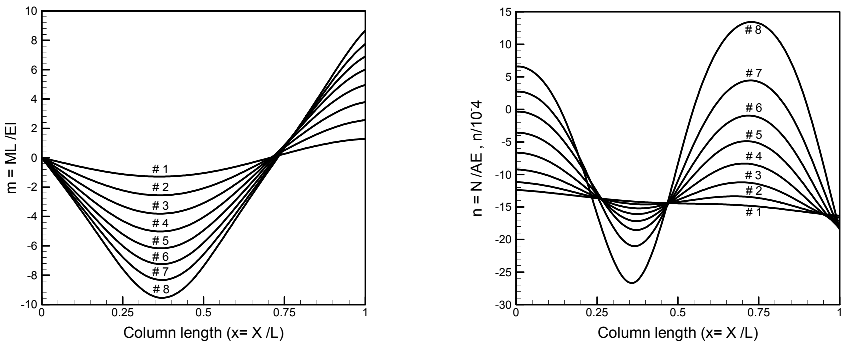
From the strong nonlinearity of the governing equations, the DQ method is used for discretization, a process which produces a set of nonlinear equations that could be solved via Newton's iteration method. The DQ discretized form of Eq. (9) can be written as

$$\sum_{j=1}^{n}c_{ij}^{(2)}\theta_j+\bar{P}R_i\sin\theta_i+\bar{H}R_i\cos\theta_i+\bar{q}\,\bar{x}_i\sin\theta_i=0$$
$$R_i=1-\frac{1}{\lambda^2}(\bar{P}\cos\theta_i-\bar{H}\sin\theta_i+\bar{q}\,\bar{x}_i\cos\theta_i),2\leq i\leq(n-1) \tag{16}$$
$$\sum_{j=1}^{n}c_{ij}^{(1)}\theta_j\Big|^{i=1}=0,\ \theta_n=0$$



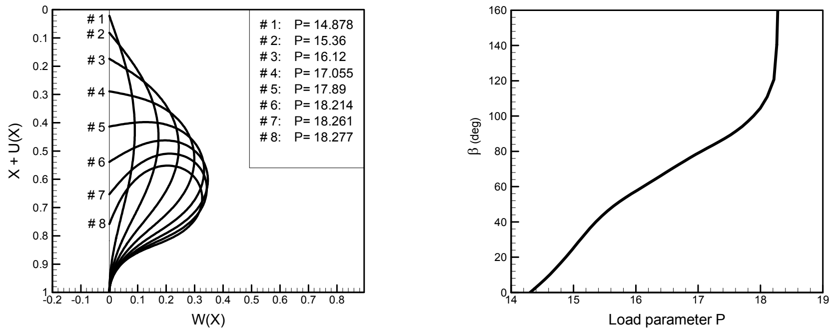
(a) (b)

Fig. 2. Post-buckling configuration for (a) $\overline{q} = 0$, (b) $\overline{q} = 5$.



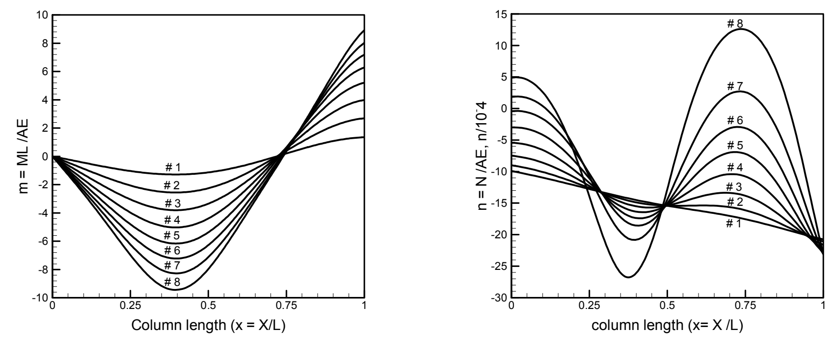
(a) (b)

Fig. 3. Variation of (a) bending moment and (b) axial force parameter for $\overline{q} = 5$.



(a) (b)

Fig. 4. Post-buckling results for $\overline{q} = 15$ (a) Column configuration, (b) Variation of β across $\overline{P}$.



(a) (b)

Fig. 5. Variation of (a) bending moment and (b) axial force parameter for $\overline{q} = 15$.

The post-buckling configuration of the column is the goal of this section, which is plotted via Eq. (16) in conjunction with Newton's iteration method. In the next figures the following symbols are used: #1: $\beta = 20^\circ$, #2: $\beta = 40^\circ$, #3: $\beta = 60^\circ$, #4: $\beta = 80^\circ$, #5: $\beta = 100^\circ$, #6: $\beta = 120^\circ$, #7: $\beta = 140^\circ$, #8: $\beta = 160^\circ$.

Post-buckling configurations for different $\bar{q}$ are shown in Figs. 2(a), 2(b) and 4(a). In addition, the characteristic curves of bending moment and axial force for different q are plotted in Figs. 3 (a), 3 (b), 5 (a) and 5 (b), respectively.

Results show that when P reaches a certain value, the small increment of P causes the great increment in beta. For example, in $\bar{q} = 15$ (see Fig. 4(b)) when P reaches approximately 18, then beta increases from 120 to 160 degree for a small increment in P. The critical points of the column must be considered. For example, in $\bar{q} = 15$ (see Fig. 5), the first relative extremum for the bending moment parameter occurs near $x = 0.4$. In addition, the axial force parameter has the relative extremum in this point that can produce critical condition for failure occurrence. The second critical condition occurs in the clamped end of the column where both bending moment and axial force have relative extremum. Figs. 2(a), 2(b) and 4(a) show that an increase in $\bar{q}$ causes a decrease in P_{cr} and vice versa, but the variation of $\bar{q}$ has a trivial effect on the magnitude of the bending moment and axial force through the column because the increment in $\bar{q}$ is accompanied by the decrement in P_{cr}.

5. Conclusion

This paper presents a formulation and a solution for post-buckling analysis of initially vertical elastic hinged-fixed columns that are subjected to terminal forces and a gravitational field. Based on geometrically nonlinear theory (considering extensibility of the column), an exact mathematical model was established. The simple but powerful numerical procedure, i.e. differential quadrature method, is employed to provide a numerical solution for initial buckling load and post-buckling configuration of the column. The present results are in good agreement with available references; the simplicity of the present method in computer programming is noticeable.

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